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# 3

## TERMINOLOGY

addition principle
circular permutation
combinatorics
factorial
multiplication principle
mutually exclusive
permutation
pigeonhole principle
recursive definition

COMBINATORICS

# COUNTING METHODS AND PERMUTATIONS



Prior learning

## PERMUTATIONS (ORDERED ARRANGEMENTS)

- solve problems involving permutations (ACMSM001)
- use the multiplication principle (ACMSM002)
- use factorial notation (ACMSM003)
- solve problems involving permutations and restrictions with or without repeated objects. (ACMSM004)

## THE PIGEON-HOLE PRINCIPLE

- solve problems and prove results using the pigeon-hole principle. (ACMSM006) AC

# 3.01 THE MULTIPLICATION PRINCIPLE

Counting is fundamental to mathematics, science, technology and our daily lives. **Combinatorics** is the mathematics of systematically counting *finite* collections. The methods of counting are most important for finite sets that produce many things to be counted.

Consider a restaurant menu that has 5 different starters, 6 different main courses and 4 different desserts. How many different three-course meals could you choose? Obviously, for each starter you could have 6 different main courses, so that makes $5 \times 6 = 30$ different combinations of starter and main course. For every one of these combinations, you could have 4 different desserts, so there would be $30 \times 4 = 120$ different three-course meals. This is an example of the **multiplication principle**.

### IMPORTANT

The **multiplication principle** says that for a choice made in two stages with $a$ ways for one part and $b$ ways for the second part, there are $a \times b$ choices altogether.

For a choice made in $n$ stages: if there are $a_1$ ways for the first part, $a_2$ ways for the second part, $a_3$ ways for the third part, and so on, then there are $a_1a_2a_3 \dots a_n$ choices altogether.

For an extended list like that above, we usually say '$a_i$ ways for the $i$th part'.

You can show a multiple product like a multiple sum, but you use $\Pi$ instead of $\Sigma$. So the product notation $\prod_{i=1}^{3} a_i$ means $a_1 \times a_2 \times a_3$.

### Example 1

A personal identification number (PIN) has 4 digits. How many PINs are possible?

#### Solution

| | |
|---|---|
| There are 4 digits in the PIN. | Number of choices $= a_1 \times a_2 \times a_3 \times a_4$ |
| Each part has 10 possible choices. | $= 10 \times 10 \times 10 \times 10 = 10\ 000$ |
| Write the answer. | There are 10 000 possible PINs. |



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When you choose coloured markers for a board game like monopoly, each person has one less marker to pick from than the person before. You usually throw a die to pick who goes first and then go around the rest of the group.

## Example 2

The markers in a board game are green, red, black, white, blue and purple. In how many different ways can 4 players pick their colours?

### Solution

| | |
|---|---|
| There are four people. | Number of choices $= a_1 \times a_2 \times a_3 \times a_4$ |
| The first person picks from 6 colours, the next from 5; so $a_1 = 6$, $a_2 = 5$, and so on. | $= 6 \times 5 \times 4 \times 3$ |
| Work out the answer. | $= 360$ |
| Write the answer. | The players can pick their colours in 360 ways. |

### IMPORTANT

The **factorial** sequence is written as $n!$ and defined as either:

$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$,

or **recursively** as $0! = 1$ and $n! = n \times (n-1)!$

You often get the pattern of numbers shown in Example **2**. This kind of pattern is so common that there is a special notation for products that go all the way down to 1. An exclamation mark is used on the end, for example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

The second method of working out the factorial sequence is a **recursive** definition. You saw in junior maths that the sequence 3, 5, 7, 9, 11, … can be given as $t_n = 2n + 1$ or recursively as $t_1 = 3$ and $t_n = t_{n-1} + 2$.

If you work downwards from $4! = 24$, then $3! = \frac{4!}{4} = \frac{24}{4} = 6$, $2! = \frac{3!}{3} = \frac{6}{3} = 2$, $1! = \frac{2!}{2} = \frac{2}{2} = 1$ and $0! = \frac{1!}{1} = \frac{1}{1} = 1$. You can also think of $0!$ as the number of ways of selecting no colours in Example **2**. There is only one way to pick none because there is only one way to leave out all of the colours.

WS

Factorial notation

## INVESTIGATION Growth rates

Some sequences of numbers increase by adding the same amount to each successive term. For example, the sequence 3, 8, 13, … increases by 5 each term. This is called **arithmetic growth**.

Another kind of growth uses powers. For example, triangular numbers like 1, 3, 6, 10, 15, … show the number of dots in triangular arrangements with sides of 1, 2, 3, … The number of dots go up according to the rule $a_n = 0.5n^2 + 0.5n$. This is called **polynomial growth** because the term is a polynomial involving $n$. This is 'faster' than increasing by a fixed amount each time.

In a geometric sequence like 3, 15, 75, … the general term is $3 \times 5^{n-1}$. The $n$ is in the exponent, so this kind of growth is called **exponential growth**. Think of some examples of exponential growth.

Compare arithmetic growth and exponential growth: which is faster?

Now think of factorial growth: $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, …

Compare **factorial growth** with exponential growth, for any power.

Consider powers of 5, compared to factorial growth: 1, 5, 25, 125, 625, …

Which gets really big first?

Choose another power, like 15. Does $n!$ ever get bigger than $15^n$?

Can you demonstrate that factorial growth always (eventually) passes exponential growth, exponential always passes polynomial growth and polynomial growth always passes arithmetic growth?

## Example 3

A coin is tossed 6 times.

**a** How many different sequences of heads and tails are there?

**b** What is the probability of guessing the exact sequence of heads and tails?

### Solution

**a** How many H/T choices are there?

Number of choices $= a_1 \times a_2 \times a_3 \times a_4 \times a_5 \times a_6$

Each toss has 2 possible outcomes: heads or tails.

$= 2 \times 2 \times 2 \times 2 \times 2 \times 2$
$= 64$

**b** Only one of the sequences is correct.

Probability $= \frac{1}{64} = 0.015625$

Write the answer.

There is a 1.5625% probability of guessing the exact sequence.

# EXERCISE 3.01 The multiplication principle

1 $5! =$

**A** 15 **B** 20 **C** 25 **D** 120 **E** 240

2 $50! =$

**A** $50 \times 49 \times 48$ **B** $2450 \times 48!$ **C** $2 \times 25!$
**D** $5! \times 10!$ **E** $5 \times 10^{10}$

3 Example 1 A password has 4 letters, with repetition allowed. How many combinations are possible?

4 A motorcycle numberplate is made up of any 2 letters followed by any 2 numbers. How many numberplates of this type are available?

5 A password has 5 letters followed by 4 numbers. If any letter of the alphabet or number (including repetitions) can be used, how many different passwords can be formed? Leave your answer in index form.

6 A witness saw most of the numberplate on a getaway car except for the first letter and the last number. How many different cars do the police need to check in order to find this car, assuming the numberplate has 3 letters and 3 numbers?

7 A certain brand of computer has a serial number made up of 10 letters followed by 15 numbers. How many computers with this type of serial number can be made? Leave your answer in index form.

8 Australia uses 4 digit postcodes and Victoria's start with 3. How many different postcodes are available in Victoria?

9 A country town has telephone numbers starting with 63 followed by any 6 other numbers from 0 to 9. How many telephone numbers are possible in this town?

10 Yasmin has 12 tops, 5 pairs of jeans and 5 pairs of shoes in her wardrobe. If she randomly chooses a top, pair of jeans and shoes, how many combinations are possible?

11 A car manufacturer produces cars in 8 different colours, with either manual or automatic gear transmission, and 4 different types of wheels. How many different combinations can it produce?

12 A company manufactures 20 000 000 computer chips. If it uses a serial number on each one that consists of 10 letters, will there be enough combinations for all these chips?

13 A manufacturer of computer parts puts a serial number on each part, consisting of 3 letters, 4 numbers, then 4 letters. The number of parts sold is estimated to be 5 million. Will there be enough combinations of this serial number to cope with these sales?

14 Example 2 Kate has chosen a pale blue, a pale green and a pale orange dress for her 3 bridesmaids. If the three colours are randomly given to each bridesmaid, how many different possibilities are there?

15 In a computer car race game, the cars that come first, second and third are randomly awarded. If there are 20 cars, how many possible combinations of first, second and third are there?

16 Jacquie only has 4 chocolates left and decides to randomly choose which of her 6 friends will receive one. How many possible ways are there in which can she give the chocolates away?

17 Three prizes are given away at a concert by taping them underneath random seats. If there are 200 people in the audience, in how many ways can these prizes be won?

18 There are 7 clients at a hairdressing salon. If 3 free haircuts are randomly given away, in how many ways could this be done?

19 A flock of 28 pelicans is fed 6 fish carcasses. If each carcass is given to a different pelican, in how many ways can this happen?

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20 Example 3 A PIN has 4 numbers. If I forget my PIN, I am allowed 3 tries to get it right. Find the probability that I get it within the 3 tries.

21 A restaurant offers 7 main courses and 4 desserts, as well as 3 different types of coffee.
- a How many different combinations of main course, dessert and coffee are possible?
- b Find the probability that I randomly pick the combination voted as the best by customers.

22 A telephone number in a capital city can start with a 9 and has 8 digits altogether.
- a How many telephone numbers are possible?
- b If I forget the last 3 digits of my friend's telephone number, how many numbers would I have to try for the correct number?

23 A password consists of 2 letters followed by 5 numbers. What is the probability that I randomly guess the correct password?

24 A bridal shop carries 12 different types of bridal dresses, 18 types of veils and 24 different types of shoes. If Kate chooses a combination of dress, veil and shoes at random, what is the probability that she chooses the same combination as her friend Jane?

25 A set of cards is numbered 1 to 100 and 2 are chosen at random.
- a How many different arrangements of ordered pairs are possible?
- b W hat is the probability that a particular ordered pair is chosen?

26 Each of 10 cards has a letter written on it from A to J. If 3 cards are selected at random, find the probability that they spell out CAB.

## Reasoning and communication

27 A city has a population of 3 500 000. How many digits should its telephone numbers have so that every person can have one?

28 Three-pin locks have three 'pins' inside the lock that are pushed into the correct position by the key to open the lock. The pins have different possible lengths that require different keys. The pins are normally made to be of different lengths, so that the key is not straight.
- a A cheap lock has five possible pin lengths. How many different keys are possible?
- b A better three-pin lock has ten possible pin lengths. How many different keys are possible?
- c Good locks have five pins with ten possible pin lengths. How many different keys are possible for these locks?

29 The genetic code of a DNA molecule consists of arrangements of four nucleotides with the bases adenine, cytosine, guanine and thymine. The bases, referred to by their initial letters A, C, G and T, are arranged in long chains but are often broken up into three-letter 'words' such as AAT for legibility. How many such words are possible?

30 Ordinary Victorian numberplates consist of three letters followed by three numbers. How many different numberplates are possible with this arrangement? Suggest why some other states changed from two letters and four numbers to this arrangement.

31 A flute manufacturer makes flutes from different materials in different sections of the flute. Different sections can be put together for different prices to suit the customer's needs. The *head joint* may be made entirely from alloy, be silver-plated, have a silver lip-plate or be solid silver. The *body* may be alloy, silver-plated or solid silver, and the *foot joint* may be alloy, silver-plated or solid silver. In addition, the foot joint may be replaced by a B-flat foot joint. How many different combinations of parts are possible?


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32 In horseracing, a quadrella requires a punter to select the winners of four particular races or 'legs' of the quadrella. A punter makes the following selections for each leg of the quadrella.

| 1st leg | 2nd leg | 3rd leg | 4th leg |
|---|---|---|---|
| Ripabuy | Roy's Joy | Fab | Cunning As |
| Run Barry Run | Did We Win | Gadzoox | Notta Brass Razoo |
| | Pedal To The Metal | Just A Con | |

How many possible quadrellas can be formed from these selections?

33 A lab rat is outside a feeding enclosure with five doors.

a In how many ways can the lab rat enter and leave the enclosure via a door?

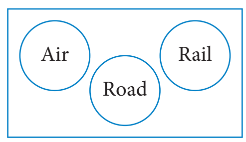
b In how many ways can the lab rat enter the enclosure through one door and leave via a different door?

# 3.02 THE ADDITION PRINCIPLE

Consider travelling from Melbourne to Alice Springs. You can fly to the Alice via Darwin, Sydney or Adelaide, or direct from Melbourne, so there are 4 ways to fly. There are 3 main ways to travel by road and 1 way by train.

The total number of ways of travelling from Melbourne to Alice Springs is $4 + 3 + 1 = 8$.

This is an example of the **addition principle**. The three transport modes are **mutually exclusive** because there is no overlap; you cannot use two methods of transport at once.




**IMPORTANT**

The **addition principle** says that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.

## Example 4

Dimitri keeps his t-shirts in a drawer and his collared shirts on hangers in his wardrobe. On Thursday morning he already has 5 shirts in the wash. He has 8 t-shirts left in his drawers and 4 collared shirts left in his wardrobe.

a How many shirts of either kind can he choose from to wear on Thursday?

b How many t-shirts and collared shirts has he got altogether?

### Solution

a Use the addition principle on clean shirts.

$$\begin{aligned}\text{Number of clean shirts} &= 8 + 4\\ &= 12\end{aligned}$$

b Add the numbers of clean and dirty shirts.

$$\begin{aligned}\text{Number of shirts} &= 12 + 5\\ &= 17\end{aligned}$$

The addition principle is often used with the multiplication principle. In some cases the multiplication part is trivial, but you can use them together to solve some quite complex problems.

## Example 5

A drawer has single socks that are identical except for colour. Three are black and the other four are white. In how many ways can you choose a matching pair?

### Solution

| | |
|---|---|
| Choose a black pair. | Number of black pairs $= 3 \times 2$<br>$= 6$ |
| Choose a white pair. | Number of white pairs $= 4 \times 3$<br>$= 12$ |
| Add the mutually exclusive pairs. | There are $6 + 12 = 18$ ways to choose a matching pair. |

The addition principle is used when you can divide a set into mutually exclusive sets. The number of elements in a union of exclusive sets is the total of the number of elements in each set. For example, the set of 5-letter words that contain B and the set of 5-letter words that do not contain the letter B are mutually exclusive sets. The sets of dogs, cats, birds, fish, and so on are mutually exclusive.

## Example 6

How many 4-letter ‘words’ can be made from the letters A, B, C, D, E, F and G if the letter B must be used and no repetition of letters is allowed?

### Solution

Imagine the B in the first position.

| | | | |
|---|---|---|---|
| **B** | | | |

6 letters could be second.

| | | | |
|---|---|---|---|
| **B** | 6 | | |

There are 5 left for the third place.

| | | | |
|---|---|---|---|
| **B** | 6 | 5 | |

There are 4 left for the last place.

| | | | |
|---|---|---|---|
| **B** | 6 | 5 | 4 |

Write the number of ‘words’ with B first.

Number with B first $= 6 \times 5 \times 4$
$= 120$

Do the same for B in the second position.

| | | | |
|---|---|---|---|
| 6 | **B** | 5 | 4 |

Write the number with B in the second position.

Number with B 2nd $= 6 \times 5 \times 4$
$= 120$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| | |
|---|---|
| Do the same for B in the third position. | $\boxed{6}\boxed{5}\boxed{\mathbf{B}}\boxed{4}$ |
| Write the number with B in the third position. | Number with B 3rd $= 6 \times 5 \times 4$<br>$= 120$ |
| Do it for B in the last position. | $\boxed{6}\boxed{5}\boxed{4}\boxed{\mathbf{B}}$ |
| Write the number with B in the last position. | Number with B last $= 6 \times 5 \times 4$<br>$= 120$ |
| Write the necessary condition. | B first, B 2nd, B third and B last are mutually exclusive. |
| Use the addition principle. | Total number of words containing B<br>$= 120 + 120 + 120 + 120$<br>$= 480$ |
| Write the answer. | The number of 4-letter 'words' containing B that can be made without repetition from the letters A, B, C, D, E, F and G is 480. |

You could also do Example **6** by saying that B could go in any of the 4 places, and then there would be choices of 6, 5 and 4 letters for each of the other places, so the total number would be $4 \times 6 \times 5 \times 4 = 480$.

## EXERCISE 3.02 The addition principle

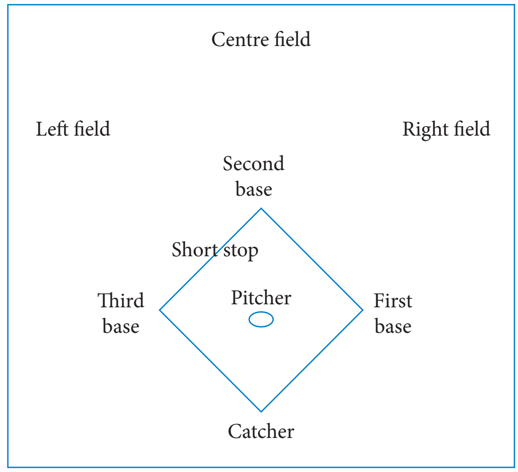
1 Example 4 An Indian restaurant has 6 chicken dishes, 8 beef dishes, 5 pork dishes, 7 lamb dishes and 10 vegetarian dishes, including 6 Vegan dishes. How many different dishes are there?

2 Meg has done her research and knows that the tablet deals from three manufacturers have enough memory and cloud access for her intended use. The first maker has only one tablet in her price range, the second has four and the third has only two. How many tablets can she pick from?

3 On a holiday in north Queensland, Jack had to decide between 3 tours of the Atherton tableland, 4 reef tours, a hot-air balloon trip and a trip on a steam train for his last day before flying home to Melbourne. How many choices did he have?

4 Example 5 Marnie has 3 red, 4 green and 2 blue tops, that she can match with her 2 red, 3 green and 2 blue skirts. The tops and skirts are all different. How many ensembles with the same colour top and skirt can she make?

5 A baseball team has 9 players. Four of the players are not very good as pitcher, short stop or catcher, so are never put in these positions. Two are good pitchers, two are good short stops, but only one is a good catcher. One of the good pitchers or short stops is always put on first base. The other fielding positions are second base, third base, left field, centre field and right field. In how many ways can they arrange the team for fielding?




6 A CD rack has three spare slots. Max has five reggae CDs and four heavy metal CDs.

a In how many ways can Max fill the three slots with CDs from the same music style?

b In how many ways can Max fill the three slots if any CD may be used?

7 Example 6 How many different 4-letter 'words' can you make from the letters of the word PARKING if no repetition is allowed and:

a you can use any letters

b you must use A

c you must use I

d you must use exactly one vowel

e you must use both vowels

f you must use at least one vowel.

8 How many 'words' can be made from the letters of the word CATERING according to each of the following rules?

a Any four letters may be used, without repetition.

b Any four letters may be used, with repetition permitted.

c One vowel and three consonants must be used in four-letter words, without repetition.

d Two vowels and two consonants must be used in four-letter words, without repetition.

e Three vowels and one consonant must be used in four-letter words, without repetition.

f At least one vowel and at least one consonant must be used, without repetition, in four-letter words.

g At least two vowels and at least two consonants must be used, without repetition, in six-letter words.

9 a How many arrangements of five different letters can be made using each of the letters in the word BEGAN?

b How many of the arrangements in part **a** begin with either A or N?

10 Each of the four digits 0, 4, 5 and 9 are used to form integers, which cannot start with 0.

a How many numbers can be formed?

b How many of the numbers in part **a** end in 0?

c How many end in 4?

d How many are even?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e How many of the numbers are odd?

f How many of the numbers are divisible by 5?

11 How many 'words' can be made from the letters of the word MATHEMATICS, according to each of the following rules?

a Four-letter words are to be made, without repetition.

b Five-letter words are to be made, without repetition.

c Five-letter words with one or two vowels are to be made, without repetition.

d Four-letter words containing the letter M are to be made, without repetition.

e Five-letter words containing the letter M are to be made, without repetition.

12 How many different arrangements can be made with the five letters E, F, G, H and I if there are three different letters in each arrangement, provided that, if an arrangement starts with a consonant, the second letter must be a vowel?

13 How many different four digit numbers can be formed with the digits 1, 4, 6, 8:

a if no digit occurs more than once in any number?

b if there are no restrictions?

## Reasoning and communication

14 A theme park has two major attractions: Underwater World and Future Land. People can go between Underwater World and Future Land by water ferry in three different ways, and by light rail in two ways.

a In how many ways can people go between Underwater World and Future Land by either water ferry or light rail?

b In how many ways can people go by water ferry and return by light rail?

c In how many ways can people go by one of the two forms of transport and return by the other?

Alamy/David Wall

15 A restaurant has breads, starters, main courses, desserts and cheese courses. There are 3 breads, 5 starters, 6 mains, 4 desserts and 2 cheese platters. How many three course meals including a course before the main course, a main course and dessert or cheese can you have?

# 3.03 THE PIGEONHOLE PRINCIPLE

The **pigeonhole principle** has many uses. Think of post office boxes.

If the post office has more letters than boxes, at least one of the boxes must get more than one letter.

If you count off boxes and objects in matching pairs, then you have some objects left over, so they must go into boxes that are already occupied. Dirichlet first stated the principle in 1834, so it is sometimes called Dirichlet's box principle.


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**IMPORTANT**

**The pigeonhole principle**

If $m$ objects are placed in $n$ boxes, where $m > n$, then at least one box must contain more than one object.

## Example 7

If you have 20 married couples, how many people must you pick to make sure you have at least one couple?

### Solution

| | |
|---|---|
| Set up your boxes. | Make 20 boxes, one for each couple. |
| Put people into the boxes. | You *can* choose one from each couple to put in the boxes. |
| Choose another person. | This person must be married to one of the first 20. |
| Write the answer. | 21 people are needed to make sure you have at least one couple. |

When you use the pigeonhole principle, you work through the worst possible case to find the minimum requirement to make sure that something happens.


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## Example 8

A basket of fruit contains apples, bananas and oranges. What is the smallest number of fruit that you should have so that there will be at least 4 apples or 5 bananas or 2 oranges in the basket?

### Solution

| | |
|---|---|
| Make a box for each fruit. | Make an apple box, a banana box and an orange box. |
| Work through the worst case. | Put in 3 apples, 4 bananas and 1 orange = 8 fruit |
| Put one more fruit in. | Whatever it is, one box will be full. |
| Write the answer. | 9 pieces of fruit will make sure there are at least 4 apples or 5 bananas or 2 oranges. |

Notice that in Example **8** you need

$(4 - 1) + (5 - 1) + (2 - 1) + 1 = 4 + 5 + 2 - 3 + 1 = 9$ fruit

to be certain of getting the required number of one fruit. This is one more than the sum of the numbers of fruit minus the number of types. It illustrates a stronger form of the pigeonhole principle.

**IMPORTANT**

**The pigeonhole principle (strong form)**

For positive integers $m_1, m_2, m_3, \ldots, m_n$ associated with $n$ boxes numbered 1 to $n$, if $m_1 + m_2 + m_3 + \cdots + m_n - n + 1$ objects are placed in the boxes, then at least one box must contain at least $m_i$ objects.

**Proof**

If you assume that the strong form is not true, then the $i$th box would contain a maximum of $m_i - 1$ objects, so the total number of objects would be a maximum of

$$(m_1 - 1) + (m_2 - 1) + (m_3 - 1) + \cdots + (m_n - 1) = m_1 + m_2 + m_3 + \cdots + m_n - n$$

which is less than the total number of objects. This is a contradiction, so it must be true. **QED**

In simple applications of the strong principle, the size of each box will often be the same. It then becomes 'if $nm - n + 1$ objects are placed in $n$ boxes, then at least one box contains at least $m$ objects'.

## Example 9

In a group of forty people, what is the largest number you can be certain will have their birthday on the same day of the week?

### Solution

| | |
|---|---|
| The worst case is when there are (nearly) the same numbers for each day. | Assume there are $m$ people with birthdays on at least one of the days. |
| Write down $m_i$ and $n$. | $n = 7$ and $m_i = m$ for $i = 1$ to 7. |

| | |
|---|---|
| Use the strong version. | Number $= m_1 + m_2 + \cdots + m_7 - n + 1$ |
| Substitute in the values. | $40 = 7m - 7 + 1$ |
| Solve for $m$. | $m = 6\frac{4}{7}$ |
| Use the fact that $m$ is an integer. | The integer part of $m = 6$. |
| Write the answer. | There must be at least 6 people with their birthday on the same day of the week. |

You could do Example **9** by distributing the people evenly between the days of the week to use 35 people, so the next person from the 40 people must make 6 on at least one day. This is really the same thing as using the principle.

# EXERCISE 3.03 The pigeonhole principle

## Concepts and techniques

1 Example 7 From a standard pack of cards, what is the smallest number needed to be sure to get:
   a two of the same suit
   b two of the same rank (two 10s, two kings, etc.)

2 What is the smallest number of socks you need to pick from a drawer with 10 single green and 10 single red socks to be certain of getting a matching pair?

3 How many people must you have at a party to be sure that at least two people will have the same birthday?

4 How many words from the first chapter in a book do you need to read to be sure of having two words starting with the same letter?

5 Example 8 A box contains mixed spare USB cables. How many cables must be in the box to be certain of having at least 6 standard cables or 5 standard to mini cables or 3 standard to micro cables or 2 standard to printer cables or 4 standard to female USB standard extension cords?

6 A citrus tart can use the juice and pulp of lemons, oranges, limes or finger limes. It requires one of the following: 4 lemons, 3 oranges, 5 limes or 8 finger limes. How many lemons, oranges, limes and finger limes need to be in a mixed bag to be certain of having enough of one kind for the recipe?

7 When David goes on holiday he likes to read a series of books from one author. Some authors take longer to read than others. He thinks he would be able to read ten of the 'Number one detective agency' books but only half as many of the 'Sunday philosophy club' series. In the same time he would be able to read six 'Rebus' novels but only three of the 'Smiley' books. If he borrowed books from a friend who had lots of books from these series, how many would he need to borrow to be certain of having enough of at least one series?

8 Example 9 There are over 33 000 undergraduate students at Monash University and there is at least one from each of the 196 countries in the world. What is the largest number of students that you can be certain must be from the same country?

9 People can have green, blue, brown, hazel or grey eyes. How many people do you need in a group to be certain of having at least 4 with eyes of the same colour?

### Reasoning and communication

10 Show that from any 101 different integers from the numbers 1 to 200, at least one is double another.

11 Show that from any 135 different integers from the numbers 1 to 200, at least 1 must be triple another.

12 Show that at least one integer must be greater than or equal to the average of a group of integers.

13 Show that in any group of 6 people there must be three people who are either mutual friends or mutual strangers.

14 Show that there are at least two people with the same number of friends in any group (more than 1), assuming that if you are friends with someone, they are also friends with you.

## 3.04 SIMPLE PERMUTATIONS

Eat, ate, eta and tea all mean something different. 'The cup split, spilt tea and was thrown away' would be wrong if you misspelt *split* and *spilt*. In a group of letters that spell words, the order is very important. When you are arranging groups of any type, it is important to be clear whether the order is important or not.

In how many different orders could we arrange the letters of the word *spilt*?

| | | | | | |
|---|---|---|---|---|---|
| Think of the letters filling up five boxes in a row. | | | | | |
| There are 5 to pick from for the first box. You could pick p. | p | | | | |
| If p is gone, there are 4 left (silt). You could pick l. | p | l | | | |
| If p and l are gone, there are 3 left (sit). You could pick i. | p | l | i | | |
| If p, l and i are gone, you could pick s from the 2 left (st). | p | l | i | s | |
| If p, l, i and s are gone, there is only one left. You must pick t. | p | l | i | s | t |
| Write the number of choices for each box. | 5 | 4 | 3 | 2 | 1 |

Actually, it doesn't make any difference what letters you choose, the number of choices *always* goes 5, 4, 3, 2 and 1.

By the multiplication principle, the total number of orders = $5 \times 4 \times 3 \times 2 \times 1 = 5!$

This can be generalised as follows.

**IMPORTANT**

The number of ways of arranging $n$ **objects in an ordered list** is given by

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

## Example 10

Eight school ID cards were dropped on the floor. In how many possible orders can the cards be picked up?

### Solution

This is just an ordered list. You can visualise the boxes or use the rule.

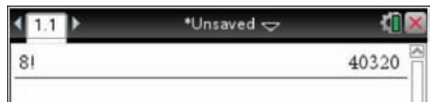
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

Number of arrangements = 8!

**TI-Nspire CAS**

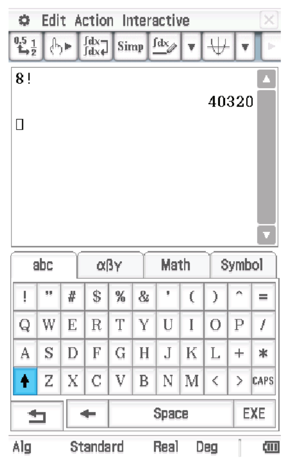
Use a calculator page.

Use the [?!▸] button on the right to get !




**ClassPad**

Simply enter 8! and tap [EXE] or press [EXE]. The factorial is found by pressing [Keyboard] and tapping [abc], then ⬆. [!] is shown in the top left of the keyboard.




Write the answer.

There are 40 320 orders in which the ID cards can be picked up.

There will be occasions when you will want to choose only some of a group of objects in order, rather than making a list of all of them.

**IMPORTANT**

A **permutation** is an arrangement of some or all of a set of symbols or objects in a particular order.

The order of objects in a permutation is important. For the three letters A, B and C there are $3 \times 2 \times 1 = 6$ different possible orders, so there are 6 permutations of the three letters A, B and C. They are ABC, ACB, BAC, BCA, CAB, and CBA.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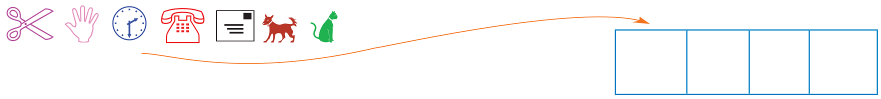
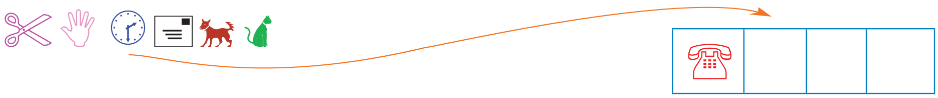
In how many ways can four objects from a set of 7 objects be arranged in a row?

We can think of the objects filling up four boxes placed in a row.

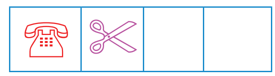


The first one chosen can be any of the 7 objects.

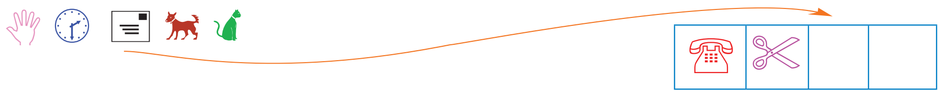
But now there are only 6 left.



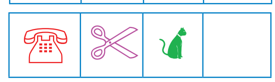
The second one chosen can be any of the 6 left.



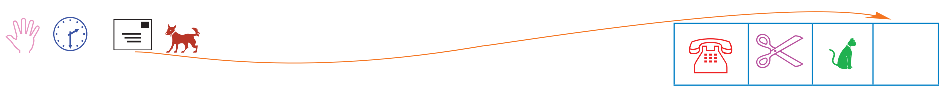
But now there are only 5 left.

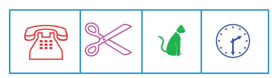


The third one chosen can be any of the 5 left.



But now there are only 4 left.





The last one can be any of the remaining 4.

Write the number of choices each time in the boxes. $\boxed{7}\times\boxed{6}\times\boxed{5}\times\boxed{4}$

By the multiplication principle, the total number of choices is $7 \times 6 \times 5 \times 4 = 840$.

Notice that $7\times6\times5\times4 = \dfrac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1} = \dfrac{7!}{3!} = \dfrac{7!}{(7-4)!}$.

You can obviously generalise this to give the following rule.

**IMPORTANT**

The number of permutations of $r$ objects from $n$ objects is written as ${}^nP_r$, which is calculated using the formula ${}^nP_r = \dfrac{n!}{(n-r)!} = n\times(n-1)\times(n-2)\times\cdots\times(n-r+1)$.

You may prefer to use boxes instead of the formula to calculate permutations, at least for small numbers of items. For example, if you had to select a committee of three people from Anne, Bob, Cathy, David and Elizabeth. You can also use your CAS calculator.

## Example 11

How many different arrangements of 5 objects in a straight line can you make from 20 objects?

### Solution

You can use 5 boxes.

| | | | | |
|---|---|---|---|---|
| 20 | 19 | 18 | 17 | 16 |

Write the formula. $\quad {}^nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$

Work out the last number. $\quad n - r + 1 = 20 - 5 + 1 = 16$

Substitute in the values. $\quad {}^{20}P_5 = 20 \times 19 \times 18 \times 17 \times 16$

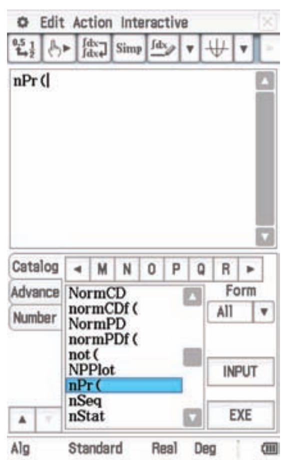
Calculate the answer. $\quad = 1\,860\,480$

**TI-Nspire CAS**

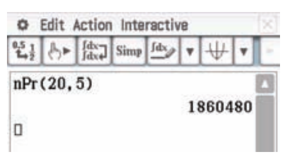
Use a Calculator page.
Use menu, 5: Probability and 2: Permutations to get nPr(). Type in the 20, 5 and press enter.


**ClassPad**

For ${}^nP_r$ press Keyboard and tap ▼ to access the list of functions. Tap **N** and scroll down to tap **nPr(**. Tap ▲ to return to the previous screen.




Enter the values of $n$ and $r$ after the bracket, separated by a comma , and close the bracket with ). Tap EXE or press EXE.



## Example 12

An anagram is an arrangement of given letters in a particular order. How many anagrams of COMFREY are there?

### Solution

An anagram is a permutation of letters. Number of anagrams $= {}^7P_7$

Calculate the answer. $= 7!$

$= 5040$

WS Permutation calculations

# EXERCISE 3.04 Simple permutations

WS Permutations

## Concepts and techniques

1 ${}^nP_0 =$ A $0$ B $1$ C $n$ D $n^2$ E $n^2 - n$

2 ${}^nP_n =$ A $0$ B $1$ C $n!$ D $n^2$ E $n^2 - n$

3 ${}^nP_1 =$ A $0$ B $1$ C $n$ D $n^2$ E $n^2 - n$

4 ${}^nP_2 =$ A $0$ B $1$ C $n$ D $n^2$ E $n^2 - n$

5 Example 10 In how many ways can each of the following numbers of different objects be placed in an ordered list?
a 5 b 9 c 10 d 4 e 12

6 A family has 7 people living in the house: Mum and Dad, Grandpa and Nan, Peter, Mary and Tom. In how many different orders can their individual photos be lined up on the hall table?

7 The alphabet is normally given in the order a, b, c, … How many different orders are possible? Give your answer in an appropriate notation.

8 Example 11 Calculate the following values.
a ${}^3P_2$ b ${}^5P_3$ c ${}^7P_2$ d ${}^9P_8$ e ${}^8P_4$
f ${}^5P_5$ g ${}^{11}P_2$ h ${}^{10}P_5$ i ${}^{85}P_3$ j ${}^{14}P_5$

9 Find $n$ if:
a ${}^nP_4 = 56 \times {}^nP_2$ b $12 \times {}^nP_3 = {}^{3n}P_2$ c $8 \times {}^nP_2 = 7 \times {}^{n+1}P_2$
d $9 \times {}^{n+1}P_3 = 10 \times {}^nP_3$ e ${}^{n+1}P_4 = 18 \times {}^nP_2$ f $56 \times {}^nP_2 = {}^nP_4$

10 Four cars enter a carpark and four parking spaces are available. In how many ways can the cars occupy the parking spaces?

11 Example 12 In how many ways can the letters of the word FIGURES be arranged?

12 A school orchestra has to play six pieces at a concert. The conductor is trying to decide the order of the pieces. How many choices does she have?

13 How many four-digit odd numbers can be formed with the digits 1, 4, 5 and 7:
a if repetitions are not allowed? b if repetitions are allowed?

14 A combination lock is opened by turning the dial to three *different* numbers in succession. How many different combinations are possible if the numbers run from:

a 0 to 9? b 0 to 19? c 0 to 29?

### Reasoning and communication

15 a How many permutations of the letters of the word ABLE are there?
b How many permutations of the letters of the word VIABLE are there?
c What is the probability that a random permutation of the word VIABLE begins with VI?

16 Find the probability that an anagram of the word COMPUTER will start with the letter C.

17 Show that if an extra item is added to a list of $n$ items, then the number of possible orders increases by $(n - 1)!\, n^2$.

# 3.05 RESTRICTED PERMUTATIONS

Permutations may be restricted in some way, and you may have to use the addition and/or multiplication principles with permutations to solve problems.

### Example 13

How many different whole numbers can be made from the digits 0, 1, 2, 3 and 4?

#### Solution

| | |
|---|---|
| Count the 1-digit numbers. | No. of 1-digit numbers = 5 |
| The 2-digit numbers cannot start with 0. | No. of 2-digit numbers $= 4 \times {}^4P_1 = 4 \times 4 = 16$ |
| The 3-digit numbers cannot start with 0. | No. of 3-digit numbers $= 4 \times {}^4P_2 = 4 \times 6 = 24$ |
| The 4-digit numbers cannot start with 0. | No. of 4-digit numbers $= 4 \times {}^4P_3 = 4 \times 4 = 16$ |
| The 5-digit numbers cannot start with 0. | No. of 5-digit numbers $= 4 \times {}^4P_4 = 4 \times 1 = 4$ |
| The 1, 2, 3, 4 and 5 digit numbers are mutually exclusive, so add the numbers. | Total number $= 5 + 16 + 24 + 16 + 4 = 65$ |
| Write the answer. | 65 whole numbers can be made. |

In some cases, you need to think of permutations as 'blocks' collapsed into a single box to solve a problem. Then you use the multiplication principle.

## Example 14

Seven people, including three friends, are seated in a row at random. In how many ways can the three friends be seated together?

### Solution

Show the row with the friends together.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | $F_1$ | $F_2$ | $F_3$ | | |

Collapse them to a single box to make a row of 5.

| | | | | |
|---|---|---|---|---|
| | | F | | |

Write the number of permutations of the new row.

Row permutations = 5!

Show the block of 3 friends.

| | | |
|---|---|---|
| $F_1$ | $F_2$ | $F_3$ |

Write the number of permutations of the friends.

Friend permutations = 3!

Use the multiplication principle.

Total arrangements = $5! \times 3!$

Calculate the answer.

$= 120 \times 6$

$= 720$

Write the answer.

There are 720 ways in which the three friends can be seated together.

Permutations that are arrangements in a circle are called **circular permutations**. They are restricted compared with straight-line permutations because they have no starting point. For example, if there are six friends sitting at a round table, then their relative positions do not depend on where you start looking.

**Proof**

Suppose there are $X$ circular permutations.

Consider one such permutation. It can be unbent into $n$ straight permutations by cutting it before each of the objects or symbols.

Thus the number of straight permutations = $nX$.

But the number of straight permutations is $n!$

Thus $nX = n!$, so $X = \frac{n!}{n} = (n - 1)!$ **QED**

**IMPORTANT**

The number of **circular permutations** of $n$ different objects or symbols is $(n - 1)!$

You can also show that the number of circular permutations is $(n - 1)!$ by using the fact that since the symbols are in a circle, it doesn't matter which one you start with.

## Example 15

In a barn dance there are 8 boys in one circle. How many possible arrangements of the boys are there?

### Solution

| | |
|---|---|
| Use the formula. | Number of circular permutations $= (8 - 1)!$<br>$= 7!$ |
| Calculate the result. | $= 5040$ |
| Write the answer. | There are 5040 arrangements of the 8 boys. |

# EXERCISE 3.05 Restricted permutations

## Concepts and techniques

WS Circular calculations

1 The number of circular permutations of 5 objects is:
A 5 B 24 C 25 D 120 E 720

2 Example 13 How many four-digit numbers can be made using the digits 0, 3, 5, 8, 2 and 7 if:
a the same digit cannot be used twice?
b repetition is allowed?

3 Given the digits 0, 2, 4, 6 and 7 and not repeating any:
a how many 5-digit numbers can be made?
b how many numbers can be made?
c how many numbers can be made that are under 3000?

4 Example 14 How many arrangements of the letters of the word FASHION are possible if:
a the vowels must all be together?
b the vowels are not all together?
c the letter N must be last?

5 In how many ways can four different black and five different red discs be placed in a row if they are placed alternately?

6 A group of 5 boys and 5 girls line up outside a cinema. Find the number of ways that can they be arranged for each of the following cases.
a With no restriction
b If a particular girl stands in line first.
c If they alternate between boys and girls.

7 Example 15 A group of friends go into a restaurant and are seated around a circular table. Find the number of arrangements that are possible if there are:
a 4 friends b 7 friends c 8 friends d 10 friends e 11 friends

8 A merry-go-round proprietor has ten different ‘horses’ to place on the outer edge of the merry-go-round. How many different placements can be made?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9 A group of 7 people sit around a table. Write the number of ways in which they can be arranged for each of the following cases.

a With no restrictions

b If 2 people want to sit together

c If 2 people cannot sit together

d If 3 people want to sit together

10 Find the probability that if 10 people sit around a table, 2 particular people will be seated together at random.

11 Six women and six men attend a dinner at a restaurant that has round dining tables. In how many ways can the people be seated so that no two women sit together?

## Reasoning and communication

12 A string of different beads looks the same when turned over. Find the number of different arrangements possible with each of the following numbers of beads.

a 10 b 12 c 9 d 11 e 13

13 A singer has arrangements of 20 different songs from her repertoire that are suitable for a particular backing group. Only three of them can be used as the opening number, and there are another four that she would prefer to choose from for the finale. In a six-song bracket, how many different programs are possible if the middle four songs are chosen from the remaining 13 arrangements?

14 The flags of a number of countries consist of three vertical stripes in different colours. For example, France's flag, from the flagpole side, is blue, white and red. If the colours red, blue, green, white, yellow and black may be used, how many flags with three vertical stripes are possible, given that:

a the three colours must be different?

b the centre colour must be different from the end colour(s)?

15 A bookshelf is to hold 5 mathematics books, 8 novels and 7 cookbooks.

a In how many different ways could the books be arranged? (Leave your answer in factorial notation.)

b If the books are grouped in categories, in how many ways can they be arranged? (Answer in factorial notation.)

16 A minibus has 6 forward-facing and 2 backward-facing seats. If 8 people use the bus, in how many ways can they be seated:

a with no restrictions?

b if one particular person must sit in a forward-facing seat?

c if 2 particular people must sit in forward-facing seats?

17 There are ten contestants in an archery competition. Four of the contestants are women. Prizes are awarded to the top four competitors. If at least one woman finishes in the first four places, in how many ways can the top four places be filled?

Getty Images/Amwell

# 3.06 PERMUTATIONS WITH REPETITION

You will often have to deal with a permutation where some of the objects are identical. Think about anagrams of word BABY. If you write the first B as $B_1$ and the second as $B_2$, the word is $B_1AB_2Y$. Without the subscripts, you cannot tell the difference between this and $B_2AB_1Y$. In the same way, $AB_1B_2Y$ and $AB_2B_1Y$ are the same. Every permutation has two versions where the Bs are swapped. Instead of having 4! permutations there will actually only be $\frac{4!}{2}$ permutations.

What about the word NEEDLE? You would not be able to tell the difference between $NE_1E_2DLE_3$, $NE_1E_3DLE_2$, $NE_2E_1DLE_3$, $NE_2E_3DLE_1$, $NE_3E_1DLE_2$ and $NE_3E_2DLE_1$. This time the 3 Es would make $3! = 6$ versions of every permutation, so the number of permutations of the word NEEDLE would be $\frac{6!}{3!}$

The word COOEE would have two versions of the positions of the Os and two versions of the positions of the Es, so the number of permutations would be $\frac{5!}{2\times 2}$.

The permutations of the letters A, A, A, B, B, C, D, E would have $3! \times 2!$ versions of the anagram EBABACAD. These correspond to the possible orders of the 3 As and the 2 Bs. There would actually be only $\frac{8!}{3!\times 2!}$ permutations, instead of 8!.

**IMPORTANT**

The number of permutations of $n$ symbols, of which $a$ are identical, is given by $\frac{n!}{a!}$.

If there are $a$, $b$, $c$, … identical symbols, the number of permutations of $n$ symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \cdots \le n$.

## Example 16

In how many ways can the letters of the word SUCCESSES be arranged?

### Solution

Write the number of symbols.

There are 9 letters, including 4 Ss, 2 Cs and 2 Es.

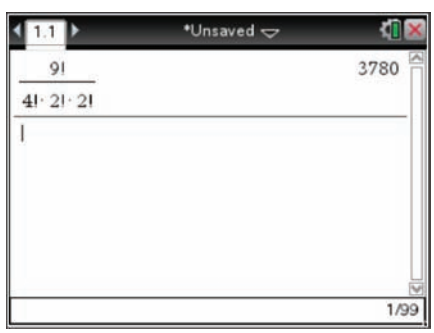
Use the formula.

$$\text{Number of permutations} = \frac{9!}{4!\times 2!\times 2!}$$

**TI-Nspire CAS**

Use the Calculator page.

Press t to get the fraction template and ?! to get !.





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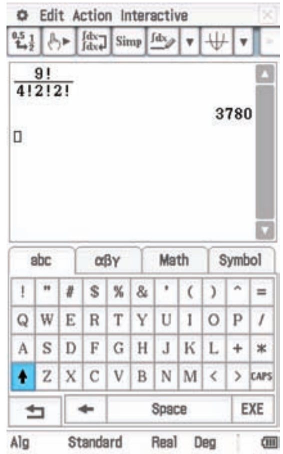
**ClassPad**

First set up a fraction using [fraction template]. (First press **Keyboard** then tap **Math1**).
Then type in the factorials from the **abc** keyboard under the **abc** tab. There is no need to enter the multiplication signs. Press **EXE** to complete the calculation.




Write the answer.

There are 3780 arrangements of SUCCESSES.

If you select only some of a group including separate symbols, then there is no simple formula for determining the number of permutations.

## EXERCISE 3.06 Permutations with repetition

### Concepts and techniques

WS Permutations with repetitions

1 Example 16 How many anagrams are there for each of the following words?

| | | |
|---|---|---|
| a CENTIPEDE | b ALGEBRA | c TELEVISION |
| d ANTARCTICA | e DONOR | f BASKETBALL |
| g GREENSLEEVES | h HOLLOWLY | i ABRACADABRA |
| j PRESTIDIGITATION | | |

2 a How many 5-digit numbers can be made using each of the digits 3, 4, 4, 5 and 6?
  b How many of the numbers are greater than 4000?
  c How many of the numbers are less than 5000?
  d If one is chosen at random, find the probability that it is less than 4000.

3 How many 5-digit numbers can be made from the digits 0, 3, 4, 4, 7?

4 How many 6-digit numbers can be made from the digits 0, 3, 4, 4, 4, 9?

5 In how many ways can 10 objects be arranged if 4 of them are identical?

6 In how many ways can 8 objects be arranged if there are 3 pairs of identical objects among them?

7 In how many ways can 20 objects be arranged if there are 2 sets of 4 identical objects among them?

### Reasoning and communication

8 How many different bracelets of 6 beads can be made from 10 different beads? Remember that a bracelet can be flipped over.

9 A competition involves 20 competitors and has three prizes. In how many ways can the prizes by awarded:
   a if no competitor can win more than one prize?
   b if any competitor can win all the prizes?
   c if no competitor can win more than two prizes?

10 One artist has 5 paintings, one has 6 and the other has 4 paintings in one room of an exhibition in a local gallery. There are doors in each of the walls of the room where they are exhibited, so the exhibition space is symmetrical around the room. In how many different ways can the artists' works be placed around the walls?

11 A coin is tossed six times. Find:
   a the number of ways it can fall
   b the number of ways you could get two heads, by finding the number of permutations of HHTTTT
   c the probability of obtaining exactly two heads from six tosses.

12 Find the probability of obtaining exactly:
   a 2 heads from 4 tosses of a coin
   b 2 heads from 8 tosses of a coin
   c 3 heads from 5 tosses of a coin
   d 6 heads from 8 tosses of a coin
   e 10 heads from 20 tosses of a coin.

# 3.07 APPLICATIONS OF COUNTING AND PERMUTATIONS

Probability problems are mostly calculated by counting the number of ways that things might happen. When solving them, you need to work out the kind of counting involved, or whether multiple methods are needed.

### Example 17

The Melbourne Cup is a handicap race with 24 horses running. The handicapper adjusts the weight that each horse carries to give each horse an equal chance of winning. A trifecta is a bet on the first three places in the correct order. Assuming that the handicapper has done the job perfectly, what is the probability of winning the trifecta on the Melbourne Cup?

### Solution

| | |
|---|---|
| The order matters, so this is a permutation. | Number of possible trifectas $= {}^{24}P_3$ |
| Work out the number. | $= 24 \times 23 \times 22$<br>$= 12\,144$ |
| Find the probability. | $P(\text{Trifecta}) = \frac{1}{12\,144}$<br>$\approx 0.000\,082$ or $0.0082\%$ |
| Write the answer. | The probability of winning the trifecta is about 0.0082%. |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When doing multiple calculations, it is often easier to do all the evaluations at the end.

### Example 18

Four couples go to a play, but their tickets have been mixed up, although the eight people are still sitting together. What is the probability that, according to the tickets, all four couples are sitting with their partners?

#### Solution

| | |
|---|---|
| Find the orders of the whole 8. | Possible orders $= {}^8P_8$ <br> $= 8!$ |
| Find the couples' orders. | Possible orders of couples $= 4!$ |
| Find the orders of each couple. | Possible orders of a couple $= 2!$ |
| Use the multiplication principle. | Possible orders with couples together $= 4! \times 2! \times 2! \times 2! \times 2!$ |
| Find the probability. | Probability(couples together) $= \dfrac{4! \times 2 \times 2 \times 2 \times 2}{8!}$ |
| Simplify. | $= \dfrac{4 \times 3 \times 2 \times 1 \times 2 \times 2 \times 2 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$ |
| Cancel and calculate the answer. | $= \dfrac{1}{7 \times 3 \times 5}$ <br> $= \dfrac{1}{105}$ <br> $\approx 0.009\,524$ |
| Write the answer. | The probability of the couples being together is just under 1%. |

## EXERCISE 3.07 Applications of counting and permutations

### Concepts and techniques

1 Example 17 A horserace meeting at a country racecourse had only 8 runners in the first race. What is the probability of winning the trifecta for this race?

2 A quadrella is a bet to pick the first four horses in a race in the correct order. What is the probability of winning a quadrella for a race with 15 runners?

3 A country race meeting only has 6 runners in the first race, 5 in the next two races and 7 in the last two races of the 5 races on the day.
What is the probability of winning a trifecta on:
a the first race?
b the last race?
An accumulator bet is one such that the winnings from one race are automatically bet on the next race. To win an accumulator bet you have to pick first place correctly on each of the races you bet on. What is the probability of winning an accumulator on
c the first 3 races?
d all of the races?

4 Example 18 What is the probability that a group of four friends will be sitting together in a row of 10 seats on an English excursion if the seats in the row are assigned at random, but all four friends are in that row?

5 Trucks taking material from construction sites to a disposal centre might be carrying soil, vegetation, rocks or old concrete. Trucks arrive at the disposal site effectively in random order. In a line of a dozen trucks waiting to enter the site, 3 are carrying soil, 3 are carrying vegetation, 3 are carrying rocks and 3 are carrying old concrete. What is the probability that the trucks carrying the same material are together in the line?

## Reasoning and communication

6 At a café there are 4 different starter courses, 7 different main courses and 5 different desserts. If the dishes are equally popular, what is the probability that the first three-course orders on Monday and Tuesday are exactly the same?

7 A number is chosen at random from those that can be made using some or all of the digits 3, 4, 5, 6 and 7 without repetition. What is the probability that it is between 400 and 4000??

8 What is the probability that a random 5-letter 'word' made from the letters of the word WICKEDLY contains a LIE? That is, contains the sequence LIE?

9 In a 'baby' competition for charity, the names of 10 teachers and photographs of them as babies are shown and students pay 50c to match the photos to the names. What is the probability of guessing correctly?


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# 3 CHAPTER SUMMARY
# COUNTING METHODS AND PERMUTATIONS

- **Combinatorics** is the mathematics of systematically counting *finite* collections.
- The **multiplication principle** states that, for a choice made in 2 stages, if there are $a$ ways for one part and $b$ ways for the second part, then there are $a \times b$ choices altogether. If a choice is made in $n$ stages, each of which has $a_i$ possibilities, then there are $a_1 a_2 a_3 \cdots a_n$ possible choices altogether.
- A **recursive** definition of a sequence finds each term from the previous one(s).
- The factorial sequence, shown as $n!$, is given by

  $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$,
  or $0! = 1$ and $n! = n \times (n-1)!$
- The $n$th term of a sequence with arithmetic growth is a fixed amount greater than the previous term.
- **Mutually exclusive** sets have no elements in common.
- The **addition principle** states that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.
- The **pigeonhole principle** states that if $m$ objects are placed in $n$ boxes, where $m > n$, then at least one box must contain more than one object.
- The **strong form** of the pigeonhole principle states that for positive integers $m_1, m_2, m_3, \ldots, m_n$ associated with $n$ boxes numbered 1 to $n$, if $m_1 + m_2 + m_3 + \cdots + m_n - n + 1$ objects are placed in the boxes, then for at least one $i$, the $i$th box contains at least $m_i$ objects.
- The number of ways of arranging $n$ **objects in an ordered list** is given by

  $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
- A **permutation** is a selection of some or all of a set of symbols or objects in a particular order.
- The number of permutations of $r$ objects from $n$ distinct objects is written as ${}^nP_r$ and is given by

$${}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).$$

- **Circular permutations** are ordered arrangements in a circle. The number of circular permutations of $n$ different objects or symbols is $(n-1)!$
- The number of permutations of $n$ symbols, of which $a$ are identical, is given by $\frac{n!}{a!}$. If there are $a, b, c, \ldots$ identical symbols, the number of permutations of $n$ symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \cdots \leq n$.

3

# CHAPTER REVIEW
# COUNTING METHODS AND PERMUTATIONS

## Multiple choice

1 Example 10 $6! =$

A 36 B 120 C 216 D 360 E 720

2 Example 11 The number of permutations of 4 symbols from 8 different symbols is

A 24 B 64 C 70 D 420 E 1680

3 Example 15 The number of ways of arranging 6 people in a circle is

A 30 B 120 C 144 D 720 E 5040

4 Example 16 The number of permutations of 6 symbols with 2 identical pairs is

A 24 B 30 C 64 D 180 E 720

5 Example 2 A new car comes in 8 different colours with 5 choices of interior trim, 3 styles of wheels and 4 different seat options. How many different cars are possible?

A 20 B 40 C 280 D 480 E 560

## Short answer

6 Example 1 An activation code for some software consists of alternating letters and numbers, with 4 letters and 3 numbers. How many different activation codes are possible?

7 Example 2 A game has 8 different coloured markers so that each player can have a different coloured marker to move around the board. In how many different ways can 6 players choose their markers?

8 Example 3 What is the probability of the first 7 babies born at the Royal Women's Hospital in Melbourne in December being boys, assuming that girls and boys are equally likely?

9 Example 5 Liza has 2 red tops, 3 pairs of red slacks, 3 blue tops, 4 pairs of blue slacks and 2 black tops. None of the items of clothing are identical. She can pair the black tops with any slacks, but otherwise needs to match the colours. How many outfits can she make up?

10 Example 6 How many 5-letter 'words' can be made from the letters of the word COSTUMING if you must use at least one vowel?

11 Example 7 A drawer contains unpaired red, blue, green, white and black socks that are identical except for their colours. How many socks must you take out to be sure of getting a matching pair?

12 Example 8 A family reunion of over 200 people has members from 4 different generations present. How many people must be in a group to be certain that there are at least 2 great-grandparents or 3 grandparents or 3 parents or 4 children?

13 Example 9 A train carriage has 60 people in it. What is the largest number of people that you can be certain will get out at one of the remaining seven stations before the end of the line?

14 Example 10 In how many different orders can 9 people in a queue be arranged?

15 Example 11 In how many ways can 5 people lined up at a buffet each choose a piece of fruit from a tray with 10 different pieces left?

16 Example 12 In how many ways can the letters of the word NUMBER be arranged?

17 Example 13 How many different whole numbers greater than 9 can be formed from the digits 0, 2, 3, 4, and 5 without repetition?

18 Example 14 In how many ways can 7 books be arranged on a shelf so that the four by the same author are together?

19 Example 16 How many anagrams of the word MACADAM are there?

20 Example 17 What is the probability of winning a trifecta in a race with 17 runners if three horses are picked at random?

## Application

21 The menu at a memorial dinner has choices of 3 different entrées, 4 different main courses and 2 different sweets. How many different three-course meals can be chosen?

22 A necklace is made from 8 natural pearls, so they are all slightly different. It also has a gold bead. The necklace is put on over the head, as it has no clasp. In how many different ways could the beads be arranged?

23 A home theatre store sells speakers from 8 different manufacturers, with a total of 92 different speakers altogether. Show that there must be at least 12 speakers of the same brand.

Qz Practice quiz

24 How many three-letter 'words' can be made with the letters of the word THREAD? The rules are that H can only go at the start or with the T to make TH, there must be at least one vowel and one consonant, and T cannot go next to D.

25 How many numbers larger than 6000 can be made from some or all of the digits 8, 6, 4, 2, 1 and 0, without repetition?